- Introduction to General Relativity and Cosmology (Winter 2016) -

Problem set No 4

Emission 25.11.16 – Digestion 16.12.16

▷ Aufgabe 1

A passenger on a merry-go-round experiences a centrifugal acceleration which is the stronger, the greater the passenger's distance from the axis of rotation. From the point of view of the passenger, the acceleration is indistinguishable from a gravitational acceleration with outwards pull.

Let x, y, z be the spatial coordinates of a point in a inertial lab system, and x', y', z' the corresponding coordinates in the rotating system of the merry-go-round. For a merry-go-round which rotates with angular frequency Ω about the Z-axis,

$$x' = \cos(\Omega t)x + \sin(\Omega t)y \,, \tag{1}$$

$$y' = \cos(\Omega t)y - \sin(\Omega t)x \,. \tag{2}$$

$$z' = z \,. \tag{3}$$

The passenger decides to use the lab clock for his time standard, i.e. $t' = t$. This may be inconvenient, since the lab clock is rotating from the passengers' perspective, but the choice would certainly not be fobidden.

(a) The Minkowski line-element, in lab coordinates, reads $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. How does ds^2 read in the merry-go-round coordinates?

(b) In the merry-go-round coordinates, the line element is not time-orthogonal, i.e. it contains terms which couple the time-differential dt to the coordinate differentials dx' and dy'. Devise a transformation $dt \to dt' = dt + \alpha(x', y')dx' + \beta(x', y')dy'$ such that the line-element becomes time-orthogonal, i.e. $ds^2 = c^2 g_{0'0'} dt'^2 + g_{x'x'}dx'^2 + g_{x'y'}dx'dy' + g_{x'y'}dy'dx' + g_{y'y'}dy'^2 + g_{z'z'}dz'^2$. Give the functions $g_{\mu'\nu'}(x', y', z', t')$ in terms of the centrifugal potential $\Phi(x', y') = -\frac{\Omega^2}{2}\left(x'^2 + y'^2\right)$.

(c) Utilizing a transformation to cylindrical coordinates $x' = \rho' \cos(\varphi')$, $y' = \rho' \sin(\varphi')$ (with z' unchanged), how is the line element expressed in these coordinates? Determine the circumference-to-radius ratio of a circle in the xa-plane which is concentric to the axis of rotation (i) in the lab-frame, (ii) in the merry-go-round frame.

(d) Would you be surprised to learn that a ride on a merry-go-round is similar to a fountain of youth?

▷ Aufgabe 2

According to a currently favored model of cosmology, on sufficiently large a spatial scale, the universe is homogeneous, isotropic and spatially flat but expands in course of time. Its metric properties are coded in the line element[1]

$$ds^2 = -c^2dt^2 + a(t)^2\left[dx^2 + dy^2 + dz^2\right] \tag{15}$$

[1] called *flat Robertson-Walker Metric*

where t has the meaning of a universal world-time, and the x^i are co-moving coordinates, which means, that galaxies' positions are described by time-independent (or just constant) x^i.

The time dependence of the scale factor $a(t)$ is determined by the Einstein equations. Depending on the model for the energy-momentum tensor (Dust, Radiation etc), one obtains

$$a(t) = (t/t_0)^q\,, \qquad 0 < q < 1\,. \tag{16}$$

with $q = \frac{2}{3}$ for a matter dominated universe, and $q = \frac{1}{2}$ for a radiation dominated universe. The parameter t_0 denotes an arbitrary moment in time in which the coordinate distance of galaxies equals their metric distance. The standard choice is "$t_0 =$ now"[2]. And since the metric is singular in the limit $t \to 0$ (all metric distances shrink to zero), the range of the time-coordinate is $0 < t < \infty$, that is space-time ends a $t = 0$. The question"what was before $t = 0$ makes no sense, since "before" does not exist.

(a) Recall: The light cones of a geometry (here Eq. (15)) are generated by null-geodesics, that is worldlines $x^i(t)$ for which $ds^2 = 0$. Determine the null geodesics which pass through the event $(ct_0, x_0, y_0, u_0$ (interpretation "Now and Here"). Given $t_0 \approx 13,8$GLj (as of 2014) – how large is the universe today? And what does this mean?

(b) A galaxy, which totay is at metric distance d always emits light of frequency ν. What would be the frequency by which this galaxy is detected today? (hint: cosmological red shift) How could you infer d from you data?

[2]which is equivalent $t_0 =$ age of the universe