2S–2P splitting in muonic hydrogen

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The relation between the transition energy that we measured and the proton root mean square charge radius r_p requires detailed calculation of relativistic, QED and recoil corrections to the energies of the $2S$ and $2P$ levels, some of which are charge-radius dependent. As the proton has a magnetic moment, the hyperfine structure correction for each level has to be taken into account, as well as QED corrections to the hyperfine structure.

The main calculations directly applicable to muonic hydrogen can be found in the works of Pachucki[1–3] and Borie[4,5]. Detailed description of existing terms can be found in the review and book of Eides, Grotch and Shelyuto[6,7]. The fine structure splitting between the $2P_{1/2}$ and $2P_{3/2}$ levels is given in Refs.[5,8]. Some hyperfine corrections can be found in the same references, but more precise calculations have been performed for the $2S$ and $2P$ states by Martynenko[8,9].

Here we describe how we arrive at the equation that was used to extract the proton charge radius from the measurement. The task is not so easy as different authors use different terminology for identical terms. Moreover the dependence on the proton radius is not always explicitly given for all terms.

The main uncertainties originate from the proton polarizability, and from different values of the Zemach radius. The Zemach radius describes the first order nuclear structure correction to the hyperfine splitting, given by the finite charge and magnetic moment distribution in the proton[10].

We started from Table I in Ref.[5] as it provides the exact Dirac contribution for the main vacuum polarization contribution to the Lamb shift. All listed contributions were checked against the other references provided above whenever possible. Summing up the corrections given in the main table, we noticed that the equation provided in Ref.[5] does not exactly correspond to the values given in the table. Since these works were published, a mistake in Ref.[11] was found and corrected in [12]. We then added a few small extra terms provided in [7]. Terms that are obtained by a formula depending only on fundamental constants have been recalculated with more significant digits. We also recalculated some of the radius-dependent part from the formulas provided in[7], to have consistent numerical accuracy whenever possible.

The list of all contributions, and the final values are presented in Table 1 for the radius-independent part of the $2S-2P$ Lamb shift, and in Table 2 for the radius-dependent part. Note that there is a term called "Nuclear size correction of order $(Z\alpha)^6$" in Table 7.1 of Ref.[7], and "Finite size of order α^6" in Table I of Ref.[2], which is not included in Ref.[5]. In Ref.[2], it is provided as a proton radius independent term -0.0009 meV for $r_p = 0.862$ fm, but it actually contains both a ${r_p}^2$ and a ${r_p}^4$ contribution to the $2S-2P_{1/2}$ splitting[7]. We include the ${r_p}^2$ dependent part in Tab. 2, but we neglect the term $0.000043\,{r_p}^4 = 2.5\cdot 10^{-5}$ meV, because it is irrelevant at our level of accuracy.

From these contributions we obtain the $2S-2P_{1/2}$ Lamb shift

$$\Delta E_{LS} = 206.0573(45) - 5.2262\, r_p^2 + 0.0347\, r_p^3 \text{ meV}, \quad (1)$$

where $r_p = \sqrt{< r_p^2 >}$ is the root mean square charge radius of the proton in fm. The uncertainties in Table 1 have been added in quadrature to obtain the total uncertainty of 0.0045 meV. This value is dominated by the proton polarizability contribution of Borie[5] (#25 in Tab. 1), who took into account the scatter of the polarizability terms by several authors for her quoted uncertainty.

The $2S$ hyperfine structure depends on the Zemach radius R_Z. Here we follow Ref.[9], who uses $R_Z = 1.022$ fm [see Eq.(7) of Ref.[13]] and adopt

$$\Delta E^{2S}_{HFS} = 22.8148\,(78) \text{ meV}. \quad (2)$$

Of above's total uncertainty of 0.0078 meV, 0.0060 meV are due to the scatter of various determinations of the Zemach radius.

The fine structure is given by[8]

$$\Delta E_{FS} = 8.352082 \text{ meV}, \quad (3)$$

which compares well with Borie's value 8.352 meV.

Finally the $2P_{3/2}$ hyperfine structure is given by[8]

$$\Delta E^{2P_{3/2}}_{HFS} = 3.392588 \text{ meV}. \quad (4)$$

The $2\,P^{F=2}_{3/2}$ level shift with respect to the unperturbed $2P_{1/2}$ level is $\Delta E_{FS} + \frac{3}{8}\Delta E^{2P_{3/2}}_{HFS} = 9.6243$ meV. For comparison, Borie's value is 9.624 meV.

The level energies are thus:

$$\begin{aligned} E_{2\,S^{F=1}_{1/2}} &= \frac{1}{4}\Delta E^{2S}_{HFS} = 5.7037\,(20) \text{ meV}, \\ E_{2\,P^{F=2}_{3/2}} &= \Delta E_{LS} + \Delta E_{FS} + \frac{3}{8}\Delta E^{2P_{3/2}}_{HFS} \\ &= 215.6816\,(45) - 5.2262 r_\text{p}^2 \\ &\quad + 0.0347 r_\text{p}^3 \text{ meV}. \end{aligned} \quad (5)$$

This leads to the transition energy for the measured $2\,S^{F=1}_{1/2} - 2\,P^{F=2}_{3/2}$ transition in muonic hydrogen:

$$\begin{aligned} E_{2\,P^{F=2}_{3/2}} - E_{2\,S^{F=1}_{1/2}} = {} & 209.9779\,(49) - 5.2262\, r_\text{p}^2 \\ & + 0.0347\, r_\text{p}^3 \text{ meV}, \end{aligned} \quad (6)$$

where the uncertainty of the constant term is the quadratic sum of the uncertainty of the Lamb shift [Eq.(1)] and 1/4 of the 2S hyperfine structure [Eq.(2)]. Note that the $r_\text{p}{}^3$ term is for a "dipole" model of the proton form factor[5].

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#	Contribution	Ref.	Our selection Value	Unc.	Pachucki[1–3] Value	Unc.	Borie[5] Value	Unc.
1	NR One loop electron VP	1,2			205.0074			
2	Relativistic correction (corrected)	1–3,5			0.0169			
3	Relativistic one loop VP	5	205.0282				205.0282	
4	NR two-loop electron VP	5,14	1.5081		1.5079		1.5081	
5	Polarization insertion in two Coulomb lines	1,2,5	0.1509		0.1509		0.1510	
6	NR three-loop electron VP	11	0.00529					
7	Polarisation insertion in two and three Coulomb lines (corrected)	11,12	0.00223					
8	Three-loop VP (total, uncorrected)				0.0076		0.00761	
9	Wichmann-Kroll	5,15,16	−0.00103				−0.00103	
10	Light by light electron loop contribution (Virtual Delbrück scattering)	6	0.00135	0.00135			0.00135	0.00015
11	Radiative photon and electron polarization in the Coulomb line $\alpha^2(Z\alpha)^4$	1,2	−0.00500	0.0010	−0.006	0.001	−0.005	
12	Electron loop in the radiative photon of order $\alpha^2(Z\alpha)^4$	17–19	−0.00150					
13	Mixed electron and muon loops	20	0.00007				0.00007	
14	Hadronic polarization $\alpha(Z\alpha)^4 m_r$	21–23	0.01077	0.00038	0.0113	0.0003	0.011	0.002
15	Hadronic polarization $\alpha(Z\alpha)^5 m_r$	22,23	0.000047					
16	Hadronic polarization in the radiative photon $\alpha^2(Z\alpha)^4 m_r$	22,23	−0.000015					
17	Recoil contribution	24	0.05750		0.0575		0.0575	
18	Recoil finite size	5	0.01300	0.001			0.013	0.001
19	Recoil correction to VP	5	−0.00410				−0.0041	
20	Radiative corrections of order $\alpha^n(Z\alpha)^k m_r$	2,7	−0.66770		−0.6677		−0.66788	
21	Muon Lamb shift 4th order	5	−0.00169				−0.00169	
22	Recoil corrections of order $\alpha(Z\alpha)^5 \frac{m}{M} m_r$	2,5–7	−0.04497		−0.045		−0.04497	
23	Recoil of order α^6	2	0.00030		0.0003			
24	Radiative recoil corrections of order $\alpha(Z\alpha)^n \frac{m}{M} m_r$	1,2,7	−0.00960		−0.0099		−0.0096	
25	Nuclear structure correction of order $(Z\alpha)^5$ (Proton polarizability contribution)	2,5,22,25	0.015	0.004	0.012	0.002	0.015	0.004
26	Polarization operator induced correction to nuclear polarizability $\alpha(Z\alpha)^5 m_r$	23	0.00019					
27	Radiative photon induced correction to nuclear polarizability $\alpha(Z\alpha)^5 m_r$	23	−0.00001					
	Sum		206.0573	0.0045	206.0432	0.0023	206.05856	0.0046

Table 1: All known radius-**independent** contributions to the Lamb shift in μp from different authors, and the one we selected. We follow the nomenclature of Eides *et al.*[7] Table 7.1. Item # 8 in Refs.[2,5] is the sum of items #6 and #7, without the recent correction from Ref.[12]. The error of #10 has been increased to 100% to account for a remark in Ref.[7]. Values are in meV and the uncertainties have been added in quadrature.

Contribution	Ref.	our selection		Pachucki[2]	Borie[5]
Leading nuclear size contribution	26	−5.19745	$< r_p^2 >$	−5.1974	−5.1971
Radiative corrections to nuclear finite size effect	2,26	−0.0275	$< r_p^2 >$	−0.0282	−0.0273
Nuclear size correction of order $(Z\alpha)^6$ $< r_p^2 >$	1,27–29	−0.001243	$< r_p^2 >$		
Total $< r_p^2 >$ contribution		−5.22619	$< r_p^2 >$	−5.2256	−5.2244
Nuclear size correction of order $(Z\alpha)^5$	1,2	0.0347	$< r_p^3 >$	0.0363	0.0347

Table 2: All relevant radius-**dependent** contributions as summarized in Eides et al.[7], compared to Refs.[2,5]. Values are in meV and radii in fm.